Electronic Supplementary Materials

1 Montonicity of $\hat{g}_k$ for $k \geq k_m$ and calculation of ϵ_g

Since Theorem 1 in the main text relies on calculation of $\epsilon_g := \sup_{k\geq K+1} \left|\hat{g}(k)\right|$ and we cannot calculate $\left|\hat{g}(k)\right|$ indefinitely, using the conclusion of Proposition 1, whose proof is completed in the following sub-section, we seek to show that there exists some integer k_m so that $\left|\hat{g}(k)\right|$ monotonically decreases for integer $k \geq k_m$. By showing monotonicity, we will have deduced

$$\epsilon_g = \sup_{k_m \geq k \geq K+1} \left|\hat{g}(k)\right|, \tag{1}$$

which is readily calculated.

We notice first that

$$\left|\hat{g}(k)\right|^2 = \frac{1}{\nu^2 k^6 \left[(1+a(k))^2 + b^2(k)\right]}, \tag{2}$$

$$a(k) = \frac{1}{\nu k^4} \Re \mathcal{N}[k] \ , \quad b(k) = \frac{C_0}{\nu k^3} - \frac{1}{\nu k^4} \Im \mathcal{N}[k], \tag{3}$$

and therefore monotonic decrease of $\hat{g}(k)$ with increasing integer k is equivalent to showing that

$$(k+1)^6 \left((1+a(k+1))^2 + b^2(k+1)\right) - k^6 \left((1+a(k))^2 + b^2(k)\right) > 1 \ , \tag{4}$$

which is equivalent to showing that

$$\begin{aligned} &\frac{(k+1)^6 - k^6}{k^5} \left(1 + 2a(k+1) + a^2(k+1) + b^2(k+1)\right) \\ &> \left(2ka(k+1) - 2ka(k) + ka^2(k+1) - ka^2(k) + kb^2(k+1) - kb^2(k)\right) . \end{aligned} \tag{5}$$

It is clearly enough to ensure that

$$\begin{aligned} &\frac{(k+1)^6 - k^6}{k^5} \left(1 - 2\left|a(k+1)\right|\right) \\ &\qquad > \left(2k\left|a(k+1)\right| + 2k\left|a(k)\right| + ka^2(k+1) + kb^2(k+1)\right) . \end{aligned} \tag{6}$$

We choose

$$k_m = \max\left\{\sqrt{\frac{3}{\nu}} R, 6\nu^{-1/2}\right\} . \tag{7}$$

Using Proposition 1, the inequalities in equation (2.7) in the main text hold, and therefore,

$$\left|\mathcal{N}[k]\right| \leq \Lambda k^2 \gamma_0 \ , \quad \text{where} \quad \gamma_0 = \frac{(1 + 2.24 \times 10^{-4})}{(1 - 1.61 \times 10^{-3})\,(1 - 0.051)}, \tag{8}$$

$$\left|\Re \mathcal{N}[k]\right| \leq \frac{\Lambda R}{4\nu} \gamma_0 + (\gamma_0 - 1)\Lambda k^2 \ . \tag{9}$$

It follows at once that for $k \geq k_m$,

$$\left|ka(k)\right| \leq \frac{R\Lambda\gamma_0}{4\nu^2 k^3} + (\gamma_0 - 1)\frac{\Lambda}{\nu k} \leq \frac{\Lambda\gamma_0}{4\sqrt{\nu}3^{3/2}R^2} + \frac{(\gamma_0 - 1)\Lambda}{\sqrt{3\nu}R}, \tag{10}$$

$$\left|b(k)\right| \leq \frac{\left|C_0\right|}{\nu k^3} + \frac{\gamma_0 \Lambda}{\nu k^2}, \tag{11}$$

and

$$ka^2(k) + kb^2(k) \leq \frac{C_0^2}{\nu^2 k^5} + \frac{2|C_0|\gamma_0 \Lambda}{\nu^2 k^4} + \frac{\Lambda^2 \gamma_0^2}{\nu^2 k^3}. \tag{12}$$

Therefore, using (10)-(12), (6) is confirmed for $k \geq k_m$ if

$$6\left(1 - \frac{\Lambda\gamma_0}{18R^3} - 2(\gamma_0 - 1)\frac{\Lambda}{\sqrt{3\nu}R}\right) > \frac{\Lambda\gamma_0}{\sqrt{\nu}3^{3/2}R^2} + \frac{4(\gamma_0-1)\Lambda}{\sqrt{3\nu}R} + \frac{C_0^2}{9k_m R^4} + \frac{2|C_0|\gamma_0\Lambda}{9R^4} + \frac{\Lambda^2\gamma_0^2}{3^{3/2}\sqrt{\nu}R^3}. \tag{13}$$

In the parameter space explored, (13) was always valid. We calculated ϵ_g based on (1).

2 Determination of $\mathcal{N}[k]$ and proof of Proposition 1.

Recall that we need to calculate

$$\mathcal{N}[k] = -\frac{i\Lambda}{\nu}\left(\frac{k\sqrt{\nu}}{2}\right) F''\left(0, k\sqrt{\nu}\right) \ , \tag{14}$$

with $\alpha = k\sqrt{\nu}$, and F satisfying the Orr-Sommerfield two point boundary value problem

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 F(y;\alpha) - i\alpha R y \left(\frac{d^2}{dy^2} - \alpha^2\right) F(y;\alpha) = 0, \tag{15}$$

$$F(0;\alpha) = 0, \quad F'(0;\alpha) = 1 \ \ , \ F(1;\alpha) = 0 \quad , \ \ F'(1;\alpha) = 0. \tag{16}$$

Using vorticity, $w = \left(\frac{d^2}{dy^2} - \alpha^2\right) F$, it is clear that we may write

$$w(y) = C_1 \mathrm{Ai}(z) + C_2 \mathrm{Ai}(\omega z), \quad z = (i\alpha R)^{1/3}\left(y - \frac{i\alpha}{R}\right), \quad \text{where} \quad \omega = e^{2i\pi/3}. \tag{17}$$

It follows that

$$F(y,\alpha) = C_1 A_1(y;\alpha) + C_2 A_2(y;\alpha) + C_3 \sinh(\alpha y) + C_4 \cosh(\alpha y) \ , \tag{18}$$

where, with $z' = (i\alpha R)^{1/3}(y' - i\alpha/R)$, we define

$$A_1(y,\alpha) = \frac{1}{\alpha}\int_0^y \sinh\left(\alpha(y-y')\right) Ai(z') dy', \quad A_2(y,\alpha) = \frac{1}{\alpha}\int_0^y \sinh\left(\alpha(y-y')\right) Ai(\omega z') dy'. \tag{19}$$

It is convenient to define images of $y = 0, 1$ under the mapping $z(y)$ to be z_0, z_1 respectively and similarly the images of those points under $\omega z(y)$ to be z_2 and z_3 respectively. Calculation shows

$$z_0 = e^{-i\pi/3}\alpha^{4/3}R^{-2/3}, \quad z_1 = z_0\left(1 + \frac{iR}{\alpha}\right), \qquad z_2 = e^{i\pi/3}\alpha^{4/3}R^{-2/3}, \quad z_3 = z_2\left(1 + \frac{iR}{\alpha}\right). \tag{20}$$

It is to be noted that when $\alpha^2 >> R$, each z_0 and z_1 are large, with $\arg z_0 = -\pi/3$ while $\arg z_1 \in \left(-\frac{\pi}{3}, \frac{\pi}{6}\right)$, and indeed close to $\arg z_0$ when $\alpha >> R$. Note that $A_1'(0,\alpha) = A_1(0,\alpha) = 0 = A_2(0,\alpha) = A_2'(0,\alpha)$. Satisfying boundary conditions (16) completely determines C_1, C_2, C_3 and C_4 in (18) and hence $F(y;\alpha)$, which allows us to express

$$F''(0;\alpha) = \frac{n(\alpha)}{D(\alpha)} \ , \tag{21}$$

where

$$D(\alpha) = \alpha \left(A_2(1) A_1'(1) - A_1(1) A_2'(1) \right) \ , \tag{22}$$

$$n(\alpha) = \alpha \cosh(\alpha) N_1(\alpha) + \sinh(\alpha) N_2(\alpha) \ , \tag{23}$$

and

$$N_1(\alpha) = B_2 A_1(1) - A_2(1) B_1 \ , \quad N_2(\alpha) = B_1 A_2'(1) - B_2 A_1'(1), \tag{24}$$

$$B_1(\alpha) = \mathrm{Ai}\,(z_0)\,, \qquad B_2(\alpha) = \mathrm{Ai}\,(z_2)\,. \tag{25}$$

It is also convenient to define

$$\lambda_1 = e^{\alpha} e^{z_0^{3/2}} z_0^{1/2} \alpha^{-1} \ , \qquad \lambda_2 = e^{-\alpha} e^{-z_0^{3/2}} z_0^{1/2} \alpha^{-1}, \tag{26}$$

and integrals

$$I_1 = \int_{z_0}^{z_1} e^{-z_0^{1/2} z} \mathrm{Ai}(z) dz \ , \quad I_2 = \int_{z_0}^{z_1} e^{z_0^{1/2} z} \mathrm{Ai}(z) dz \ ,$$

$$I_3 = \omega^{-1} \int_{z_2}^{z_3} e^{z_2^{1/2} z} \mathrm{Ai}\,(z)\, dz \ , \quad I_4 = \omega^{-1} \int_{z_2}^{z_3} e^{-z_2^{1/2} z} \mathrm{Ai}\,(z)\, dz \ . \tag{27}$$

It follows from (19) that

$$A_1(1;\alpha) = \frac{1}{2\alpha} \left(\lambda_1 I_1 - \lambda_2 I_2 \right) \ , \quad A_1'(1;\alpha) = \frac{1}{2} \left(\lambda_1 I_1 + \lambda_2 I_2 \right), \tag{28}$$

$$A_2(1;\alpha) = \frac{1}{2\alpha} \left(\lambda_1 I_3 - \lambda_2 I_4 \right) \ , \quad A_2'(1;\alpha) = \frac{1}{2} \left(\lambda_1 I_3 + \lambda_2 I_4 \right). \tag{29}$$

Therefore,

$$D(\alpha) = \frac{\lambda_1 \lambda_2}{2} \left(I_2 I_3 - I_1 I_4 \right) \tag{30}$$

$$n(\alpha) = \frac{1}{2} \left[\lambda_1 e^{-\alpha} (B_2 I_1 - B_1 I_3) - \lambda_2 e^{\alpha} (B_2 I_2 - B_1 I_4) \right] \tag{31}$$

and so, using $\alpha = k\sqrt{\nu}$ and expression for $F''(0,\alpha)$ in (21), it follows from (14) that

$$\mathcal{N}[k] = -\frac{i\Lambda}{2\nu} \left(\frac{\alpha n(\alpha)}{D(\alpha)} \right) = -\frac{i\Lambda\alpha}{2\nu} \left(\frac{\lambda_1 e^{-\alpha} \left(B_2 I_1 - B_1 I_3 \right) - \lambda_2 e^{\alpha} \left(B_2 I_2 - B_1 I_4 \right)}{\lambda_1 \lambda_2 \left(I_2 I_3 - I_1 I_4 \right)} \right). \tag{32}$$

2.1 Details of the proof of Proposition 1

Recall from the main part of the paper the functions

$$H_0(z) = \exp\left[\frac{2}{3} z^{3/2} \right] \mathrm{Ai}(z), \qquad H_j(z, z_0) = \frac{d}{dz} \left[\frac{H_{k-1}(z, z_0)}{z^{1/2} + z_0^{1/2}} \right] \ \text{for} \ j \geq 1, \tag{33}$$

$$\mathcal{U}(z) = z^{-1/2} H_0(z) \ , \quad \mathcal{V}(z) = z^{-1/2} \mathcal{U}'(z), \tag{34}$$

$$H_1(z, z_0) = m \mathcal{U}'(z) + \frac{s}{2z} m^2 \mathcal{U}(z), \tag{35}$$

$$s = z^{-1/2} z_0^{1/2} \ , \quad m = (1+s)^{-1} \ , \tag{36}$$

$$H_2(z, z_0) = m^2 \mathcal{V}'(z) + \frac{3s}{2z^{3/2}} m^3 \mathcal{U}'(z) + \left(\frac{3s^2}{4z^{5/2}} m^4 - \frac{s}{z^{5/2}} m^3 \right) \mathcal{U}(z). \tag{37}$$

We will find convenient sometimes to use $H_0(z, z_0) \equiv H_0(z)$. In addition, define

$$J(\tau) = \left(1 + \left[1 + \tau^2\right]^{1/4}\right) \left(1 + \tau^2\right)^{1/8} . \tag{38}$$

We will also need the functions $k_{0,U}$, $k_{1,U}$, $k_{1,V}$, ϵ_U, $\epsilon_{U'}$, $\epsilon_{V'}$ that are defined by equations (4.39)-(4.42) in Section 4 of the main article.

Definition 1. *We define the straight line segments L_0 and L_2:*

$$L_0 := \{z : z = z_0 + t(z_1 - z_0) \ , \ t \in [0, 1]\} , \quad L_2 := \{z : z = z_3 + t(z_2 - z_3) \ , \ t \in [0, 1]\} . \tag{39}$$

Corollary 1. *$H_0(z)$ defined in (33) satisfies the following upper and lower bounds at any point on L_0 and L_2 in the regime $\alpha = k\sqrt{\nu} \geq \max\left\{\sqrt{3}R, \alpha_r\right\}$,*

$$\frac{|z|^{-1/4}}{2\sqrt{\pi}} \left(1 - k_{0,U}(z_0)\right) \leq \left|H_0(z)\right| \leq \frac{|z|^{-1/4}}{2\sqrt{\pi}} \left(1 + k_{0,U}(z_0)\right) =: C_0 |z|^{-1/4} \leq C_0 |z_0|^{-1/4} . \tag{40}$$

Proof. First we note that either on L_0 or L_2, $|z| \geq |z_0|$, since $z_1 = z_0(1 + iR\alpha^{-1})$ and $z_3 = z_2(1 + iR\alpha^{-1})$ and $z_2 = z_0^*$. Regime $\alpha = k\sqrt{\nu} \geq \max\left\{\sqrt{3}R, \alpha_r\right\}$ ensures that $|z| \geq 2$ and $\arg z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and using the definition (34) for $H_0(z)$, Lemma 12 of the main article is applicable and we obtain the given bounds, noting that $k_{0,U}(z) \leq k_{0,U}(z_0)$ (See Remark 5 in the paper). ∎

Lemma 1. *H_j defined in (33) for $j = 1, 2$ satisfy the following bounds for any point on line segments L_0 and L_2 in the regime $\alpha = k\sqrt{\nu} \geq \max\left\{\sqrt{3}R, \alpha_r\right\}$, where we define $\hat{z}_0 = z_0$ on L_0 and $\hat{z}_0 = z_2$,*

$$\left|H_1\left(z, \hat{z}_0\right)\right| , \leq C_1 |z_0|^{-7/4} , \quad \left|H_2\left(z, \hat{z}_0\right)\right| , \leq C_2 |z_0|^{-13/4} , \tag{41}$$

where

$$C_1 = \frac{3}{8\sqrt{\pi}} \left(1 + \epsilon_{U'}(z_0)\right) + \frac{1}{4\sqrt{\pi}} \left(1 + \epsilon_U(z_0)\right) , \tag{42}$$

$$C_2 = \frac{1}{\sqrt{\pi}} \left\{ \frac{27}{32} \left(1 + \epsilon_V(z_0)\right) + \frac{9}{16} \left(1 + \epsilon_{U'}(z_0)\right) + \frac{7}{8} \left(1 + \epsilon_U(z_0)\right) \right\} . \tag{43}$$

For $\left|H_1(z_3, z_2)\right|$, we also have the sharper bound

$$\left|H_1(z_3, z_2)\right| \leq C_2 |z_3|^{-7/4} \tag{44}$$

Proof. On $z \in L_0 \cup L_2$,

$$s = z^{-1/2} \hat{z}_0^{1/2} = \left(1 + iRt\alpha^{-1}\right)^{-1/2} \text{ for } t \in [0, 1] \tag{45}$$

and it is clear that $\Re s \geq 0$ and $|s| \leq 1$ in both cases, and so $|m| = |1 + s|^{-1} \leq 1$. We also note that for any $\alpha \geq 0$, for z on these straight line segments, $|z|^{-\alpha} = |z_0|^{-\alpha} \left|s\right|^{2\alpha} \leq |z_0|^{-\alpha}$. Combining (35) with bounds on $\mathcal{U}$ and $\mathcal{U}'$ in Lemma 12 of the main part, we obtain

$$\left|H_1(z, \hat{z})\right| \leq \left|\mathcal{U}'(z)\right| + \frac{1}{2|z|} \left|\mathcal{U}(z)\right| \leq \frac{3}{8\sqrt{\pi}|z_0|^{7/4}} \left(1 + \epsilon'_U(z_0)\right) + \frac{1}{4\sqrt{\pi}|z_0|^{7/4}} \left(1 + \epsilon_U(z_0)\right) , \tag{46}$$

and using (37) we have

$$\begin{aligned} \left|H_2(z, \hat{z})\right| &\leq \left|\mathcal{V}'(z)\right| + \frac{3}{2|z|^{3/2}} \left|\mathcal{U}'(z)\right| + \frac{7}{4|z|^{5/2}} \left|\mathcal{U}(z)\right| . \\ &\leq \frac{1}{\sqrt{\pi}|z_0|^{13/4}} \left\{ \frac{27}{32} \left(1 + \epsilon_V(z_0)\right) + \frac{9}{16} \left(1 + \epsilon'_U(z_0)\right) + \frac{7}{8} \left(1 + \epsilon_U(z_0)\right) \right\} \end{aligned} \tag{47}$$

For $H_1(z_3, z_2)$, we note from the definition of $H_1(z, z_0)$ that since $s = z_3^{-1/2} z_2^{1/2} = \left(1 + iR\alpha^{-1}\right)$ and $m = 1/(1+s)$ are each bounded by 1,

$$\left|H_1(z_3, z_2)\right| \le \left|\mathcal{U}'(z_3)\right| + \frac{1}{2|z_3|}\left|\mathcal{U}'(z_3)\right| \tag{48}$$

The rest follows from bounds on $\mathcal{U}$ and $\mathcal{U}'$ in Lemma 12 in the main part, the observation $|z_3| > |z_2| = |z_0|$, and the fact that each of $\epsilon_{0,U}$, $\epsilon_{1,U}$ are decreasing with $|z|$. ∎

Lemma 2. *$H_0(z_0)$ and $H_1(z_0, z_0)$ satisfy the following bound*

$$\left|\frac{H_1(z_0, z_0)}{H_0(z_0)} + \frac{1}{4z_0^{3/2}}\right| \le \frac{1}{4|z_0|^{3/2}}\epsilon_{1,0}, \tag{49}$$

where

$$\epsilon_{1,0} = \frac{3}{2}\left\{\left(1 + \frac{5}{16|z_0|^{3/2}} + \frac{8}{3}\sqrt{\pi}k_{1,U}(z_0)\right)\left(1 - \frac{5}{48|z_0|^{3/2}} - 2\sqrt{\pi}k_{0,U}(z_0)\right)^{-1} - 1\right\} \tag{50}$$

Proof. From (33) and $H_1(z, z_0) = m\mathcal{U}'(z) + \frac{s}{2z}m^2\mathcal{U}(z)$,

$$\frac{H_1(z_0, z_0)}{H_0(z_0)} = \frac{1}{8z_0^{3/2}} + \frac{\mathcal{U}'(z_0)}{2z_0^{1/2}\mathcal{U}(z_0)} \tag{51}$$

Using Lemma 12 we obtain

$$\frac{\mathcal{U}'(z_0)}{\mathcal{U}(z_0)} = -\frac{3}{4z_0}\left(1 - \frac{5}{16z_0^{3/2}} + \frac{8}{3}\sqrt{\pi}K_{1,U}(z_0)\right)\left(1 - \frac{5}{48z_0^{3/2}} + 2\sqrt{\pi}K_{0,U}(z_0)\right)^{-1}, \tag{52}$$

where $K_{1,U}$, $K_{0,U}$ are bounded, respectively, by $k_{1,U}$ and $k_{0,U}$, defined in equations (4.39)-(4.40). Therefore,

$$\left|\frac{H_1(z_0), z_0)}{H_0(z_0)} + \frac{1}{4z_0^{3/2}}\right| \le \frac{1}{4|z_0|^{3/2}}\epsilon_{1,0} \tag{53}$$

with $\epsilon_{1,0}$ as defined above. ∎

Lemma 3. *Define, for $t \in [0, 1]$,*

$$h_0(t) = \frac{-2i\alpha^2}{3R}\left(1 + \frac{iR}{\alpha}t\right)^{3/2} + \frac{2i\alpha^2}{R}, \tag{54}$$

$$h_2(t) = \frac{2i\alpha^2}{3R}\left(1 + \frac{iR}{\alpha}(1-t)\right)^{3/2} - \frac{2i\alpha^2}{3R}\left(1 + \frac{iR}{\alpha}\right)^{3/2}. \tag{55}$$

Then, for any $t \in [0, 1]$, and with $\tau = \frac{Rt}{\alpha}$,

$$\frac{d}{dt}\Re h_0(t) = \alpha\left(1 + \frac{\tau^2}{\left(\sqrt{2}\sqrt{1 + (1+\tau^2)^{1/2}} + 2\right)\left(1 + (1+\tau^2)^{1/2}\right)}\right) \ge \alpha\ , \tag{56}$$

and

$$\frac{d}{dt}\Re h_2(t) \ge \alpha\ , \tag{57}$$

implying in each case that

$$\Re h_0(t)\ , \Re h_2(t) \ge \alpha t. \tag{58}$$

Proof. On differentiation and considering the first case we have

$$\frac{d}{dt}\Re h_0(t) = \alpha \Re \left(1 + \frac{iR}{\alpha} t\right)^{1/2} \quad , \quad \frac{d}{dt}\Re h_2(t) = \alpha \Re \left(1 + \frac{iR}{\alpha}(1-t)\right)^{1/2} . \tag{59}$$

The rest follows from trigonometric simplification of

$$\Re \left(1 + i\tau\right)^{1/2} = \left(1 + \tau^2\right)^{1/4} \cos \left(\frac{1}{2} \arctan \tau\right), \tag{60}$$

and using integration with initial condition $h_0(0) = 0 = h_2(0)$. ∎

Corollary 2. *On any point on the straight line segment L_0 parameterized by $t \in [0,1]$, define,*

$$g_0(t) = \frac{2}{3} \left(z^{3/2} - z_0^{3/2}\right) + z_0^{1/2} \left(z - z_0\right). \tag{61}$$

Similarly, on any point on the straight line L_2 parametrized by $t \in [0,1]$, define

$$g_2(t) = \frac{2}{3} \left(z^{3/2} - z_3^{3/2}\right) + z_2^{1/2} \left(z - z_3\right). \tag{62}$$

Then, in either case,

$$\Re g_0(t), \Re g_2(t) \geq 2\alpha t \tag{63}$$

Proof. On L_0 using $z_0^{3/2} = -\frac{iR^2}{\alpha^2}$, $z/z_0 = 1 + \frac{iR}{\alpha} t$, we get in terms of h_0 defined in the last Lemma,

$$g_0(t) = h_0(t) + \alpha t \qquad \Rightarrow \qquad \Re g_0(t) \geq 2\alpha t. \tag{64}$$

On L_2, using $z_2^{3/2} = \frac{iR^2}{\alpha^2}$, $z/z_2 = 1 + \frac{iR}{\alpha}(1-t)$, we obtain in terms of $h_2(t)$ defined in the last Lemma

$$g_2(t) = h_2(t) + \alpha t \qquad \Rightarrow \qquad \Re g_2(t) \ \geq 2\alpha t. \tag{65}$$

∎

Lemma 4. *I_1 defined in* (27) *may be expressed as*

$$I_1 = \frac{1}{2 z_0^{1/2}} e^{-\frac{5}{3} z_0^{3/2}} \left[H_0(z_0) + H_1(z_0, z_0) + \frac{R_1}{2\sqrt{\pi} z_0^{1/4}}\right] \tag{66}$$

where R_1 satisfies the bound

$$\left|R_1\right| \leq 4\sqrt{\pi} e^{-2\alpha} \left(C_0 + C_1 |z_0|^{-3/2}\right) + 2\sqrt{\pi} C_2 |z_0|^{-3} := k_1. \tag{67}$$

In particular, we have the upper and lower bounds

$$C_{m,1} |z_0|^{-3/4} \leq \left|I_1\right| \leq C_{I,1} |z_0|^{-3/4}, \tag{68}$$

where

$$C_{I,1} = \frac{1}{2} \left(C_0 + C_1 |z_0|^{-3/2} + \frac{k_1}{2\sqrt{\pi}}\right), \quad C_{m,1} = \frac{1}{2} \left(\frac{1 - \epsilon_U(z_0)}{2\sqrt{\pi}} - C_1 |z_0|^{-3/2} - \frac{k_1}{2\sqrt{\pi}}\right). \tag{69}$$

Proof. From the definitions of I_1 and H_0 in (27) and (33), we note that

$$I_1 = \int_{z_0}^{z_1} e^{-z_0^{1/2} z - \frac{2}{3} z^{3/2}} H_0(z) dz. \tag{70}$$

On integration by parts twice, we obtain

$$I_1 = \left[\frac{e^{-z_0^{1/2} z - \frac{2}{3} z^{3/2}}}{z_0^{1/2} + z^{1/2}} \left(H_0(z) + H_1(z, z_0) \right) \right]_{z=z_1}^{z=z_0} + \int_{z_0}^{z_1} e^{-z_0^{1/2} z - \frac{2}{3} z^{3/2}} H_2(z, z_0) dz. \tag{71}$$

Therefore, we are able to write

$$I_1 = \frac{1}{2 z_0^{1/2}} e^{-\frac{5}{3} z_0^{3/2}} \left[H_0(z_0) + H_1(z_0, z_0) + \frac{R_1}{2\sqrt{\pi} z_0^{1/4}} \right], \tag{72}$$

provided, we identify $R_1 = R_{1,1} + R_{1,2}$, where

$$R_{1,1} = -\frac{4\sqrt{\pi} z_0^{3/4}}{z_0^{1/2} + z_1^{1/2}} \left[H_0(z_1) + H_1(z_1, z_0) \right] \mathrm{e}^{-g_0(1)} \ , \tag{73}$$

$$R_{1,2} = 4\sqrt{\pi} z_0^{3/4} \int_{z_0}^{z_1} \mathrm{e}^{-g_0(t)} H_2(z, z_0) dz \ . \tag{74}$$

We note that the exponents in $R_{1,1}, R_{1,2}$ are bounded by $e^{-2\alpha}$ and $e^{-2\alpha t}$, respectively. We also note the global bounds on H_1 and H_2 on any point on the straight line segment L_0 connecting z_0 to z_1 in Lemma 1. Furthermore, since $z_1 = z_0 \left(1 + \frac{iR}{\alpha} \right)$, then $\left| z_1^{-1/2} z_0^{1/2} \right| \le 1$ and $\left| 1 + z_0^{-1/2} z_1^{1/2} \right|^{-1} \le 1$. Further in the integral in $R_{1,2}$, with t parametrization of L_0, we obtain $dz = (z_1 - z_0) dt = \frac{iRz_0}{\alpha} dt$, while $\int_0^1 e^{-2\alpha t} dt \le \frac{1}{2\alpha}$, $\frac{R}{\alpha^2} = |z_0|^{-3/2}$. With this information, we readily obtain

$$\left| R_{1,2} \right| \le 2\sqrt{\pi} C_2 |z_0|^{-3}, \qquad \left| R_{1,1} \right| \le 4\sqrt{\pi} e^{-2\alpha} \left(C_0 + C_1 |z_0|^{-3/2} \right) \ , \tag{75}$$

from which the first statement of the Lemma follows. The second statement follows from the first after some algebraic manipulation. ∎

Remark 1. *Note that for large α, k_1 becomes small and approaches zero. The point of the above Lemma is to show precise bounds when α is some finite number, and therefore makes it precise how large is large.*

Lemma 5. *I_4 defined in* (27) *may be expressed as*

$$\omega I_4(z) = -\frac{e^{-z_2^{1/2} z_3 - \frac{2}{3} z_3^{3/2}}}{z_2^{1/2} + z_3^{1/2}} \left(H_0(z_3) + H_1(z_3, z_2) - \frac{1}{2\sqrt{\pi} |z_3|^{1/4}} R_4(z) \right), \tag{76}$$

where R_4 satisfies the bound

$$\left| R_4 \right| \le \sqrt{\pi} J \left(\frac{R}{\alpha} \right) \left\{ e^{-2\alpha} \left(C_0 + C_1 |z_0|^{-3/2} \right) + C_2 |z_0|^{-3} \right\} =: k_4, \tag{77}$$

and $J(\tau)$ is defined in (38). *In particular I_4 satisfies the lower bound*

$$\left| \exp\left[z_2^{1/2} z_3 + \frac{2}{3} z_3^{3/2} \right] I_4 \right| \ge \frac{C_{m,4}}{J\left(\frac{R}{\alpha} \right)} |z_0|^{-3/4} \ , \tag{78}$$

where

$$C_{m,4} = \frac{1-\epsilon_U(z_0)}{2\sqrt{\pi}} - C_1|z_0|^{-3/2} - \frac{1}{2\sqrt{\pi}}k_4. \tag{79}$$

Proof. Using (20), since $z_2^{3/2} = \frac{i\alpha^2}{R}$ and $z_3 = z_2\left(1+\frac{iR}{\alpha}\right)$, it follows from (27) that

$$I_4 = \omega^{-1}\int_{z_2}^{z_3} \exp\left[-z_2^{1/2}z - \frac{2}{3}z^{3/2}\right] H_0(z)dz. \tag{80}$$

On integration by parts twice, as for I_1, and using the straight line segment L_2 for integration, where $dz = (z_2 - z_3)dt = -\frac{iR}{\alpha}z_2 dt$, we obtain

$$\omega \exp\left[z_2^{1/2}z_3 + \frac{2}{3}z_3^{3/2}\right] I_4 = -\frac{H_0(z_3)+H_1(z_3,z_2)}{z_3^{1/2}+z^{1/2}}\left(1 - \frac{1}{2\sqrt{\pi}|z_3|^{1/4}}R_4\right), \tag{81}$$

where

$$\begin{aligned} R_4 = {} & \sqrt{\pi}z_3^{1/4}\left(1+z_3^{1/2}z_2^{-1/2}\right)\left(H_0(z_2)+H_1(z_2,z_2)\right)e^{-g_2(1)} \\ & + \frac{2iR}{\alpha}\sqrt{\pi}z_2^{3/2}z_3^{1/4}\left(1+z_3^{1/2}z_2^{-1/2}\right)\int_0^1 e^{-g_2(t)}H_2\left(z_3+t(z_2-z_3),z_2\right)dt, \end{aligned} \tag{82}$$

with g_2 as defined in (62). Thus the exponential terms are bounded by $e^{-2\alpha}$ and $e^{-2\alpha t}$, respectively. Using bounds on H_j from Lemma 1 for any point $z \in L_2$, and noting $\frac{2R}{\alpha}\int_0^1 e^{-2\alpha t}dt \le \frac{R}{\alpha^2} = |z_0|^{-3/2}$, the first statement in the Lemma follows very much like the previous Lemma, except that we have an algebraic factor of

$$\left|\left(1+z_3^{1/2}z_2^{-1/2}\right)\left(\frac{z_3}{z_2}\right)^{1/4}\right| \le J\left(\frac{R}{\alpha}\right).$$

The second part of the Theorem clearly follows from the first on algebraic manipulation where we use $\frac{z_3}{z_2} = 1+\frac{iR}{\alpha}$ and $|z_2| = |z_0|$. ∎

Remark 2. *Since*

$$\frac{2}{3}z_2^{3/2} - \frac{2}{3}z_3^{3/2} + z_2^{1/2}(z_2 - z_3) = g_2(1), \tag{83}$$

and $\Re g_2(1) \ge 2\alpha$*, while* $\Re z_2^{3/2} = 0$*, it follows that* $\left|\exp\left[-\frac{2}{3}z_3^{3/2} - z_2^{1/2}z_3\right]\right| \ge e^{2\alpha}$*, and the lower bounds in the previous Lemma show that* I_4 *is exponentially large in* α*. This exponentially large lower bound for* I_4 *for* α *large is significant, as it allows massive simplification of* $\mathcal{N}(k)$ *as we shall see shortly.*

Lemma 6. I_2 *and* I_3 *defined in* (27) *satisfy the following bounds*

$$|I_2| \le C_0 R^{1/2}, \qquad \left|\exp\left[-z_2^{1/2}z_3 + \frac{2}{3}z_3^{3/2}\right] I_3\right| \le C_0 R^{1/2}. \tag{84}$$

Proof. We take the straight line path L_0 connecting z_0 to z_1 in I_2 in (27) and obtain

$$\exp\left[-\frac{1}{3}z_0^{3/2}\right] I_2 = (z_1 - z_0)\int_0^1 e^{-\hat{g}_0(t)}H_0\left(z_0+t(z_1-z_0)\right)dt, \tag{85}$$

where

$$\hat{g}_0(t) = -tz_0^{3/2}\left(z_1/z_0 - 1\right) + h_0(t) = -\alpha t + h_0(t), \tag{86}$$

and h_0 is defined in Lemma 3, from which we can conclude that since $\Re h_0 \geq \alpha t$, we must have

$$\Re \hat{g}_0 \geq 0 \ , \quad \text{implying} \quad \left| e^{-\hat{g}_0(t)} \right| \leq 1. \tag{87}$$

Using global bounds on H_0 in Lemma 1, $z_0^{3/2} = -\frac{i\alpha^2}{R}$ and $\frac{R}{\alpha} = |z_0|^{-3/4} R^{1/2}$, we obtain

$$\left| I_2 \right| \leq C_0 |z_0|^{3/4} \frac{R}{\alpha} = C_0 \sqrt{R}. \tag{88}$$

For I_3 defined in (27), again using a straight line path of integration $z = z_3 + t(z_3 - z_2)$ and $z_2^{3/2} = \frac{i\alpha^2}{R}$, $z_3/z_2 = 1 + \frac{iR}{\alpha}$, we obtain

$$e^{-z_2^{1/2} z_3 + \frac{2}{3} z_3^{3/2}} I_3 = (z_3 - z_2) \int_0^1 e^{-\hat{g}_2(t)} H_0 \left(z_3 + t(z_2 - z_3) \right) dt \ , \tag{89}$$

where in this case

$$\hat{g}_2(t) = -t z_2^{3/2} \left(1 - \frac{z_3}{z_2} \right) + h_2(t) = -\alpha t + h_2(t), \tag{90}$$

with h_2 defined in Lemma 3. Using that Lemma, $\Re \hat{g}_2(t) \geq 0$, implying

$$\left| e^{-\hat{g}_2(t)} \right| \leq 1 \ . \tag{91}$$

Using bounds on $H_0(z)$ on the line segment L_2 in Lemma 1 and $dz = -\frac{iRz_2}{\alpha} dt$, we obtain

$$\left| e^{-z_2^{1/2} z_3 + \frac{2}{3} z_3^{3/2}} I_3 \right| \leq C_0 |z_2|^{3/4} \frac{R}{\alpha} = C_0 R^{1/2}. \tag{92}$$

∎

Lemma 7. *Define*

$$\hat{I}_1 = \frac{2 z_0^{1/2} I_1}{H_0(z_0)} e^{\frac{5}{3} z_0^{3/2}}. \tag{93}$$

Then,

$$\left| \hat{I}_1 - 1 + \frac{1}{4 z_0^{3/2}} \right| \leq \hat{C}_1 |z_0|^{-3}, \tag{94}$$

where

$$\hat{C}_1 = \frac{|z_0|^3 k_1}{1 - \epsilon_U(z_0)} + \frac{1}{4} |z_0|^{3/2} \epsilon_{1,0}. \tag{95}$$

Proof. Using (93) in Lemma 4 we have

$$\hat{I}_1 - 1 + \frac{1}{4 z_0^{3/2}} = \left(\frac{H_1(z_0, z_0)}{H_0(z_0)} + \frac{1}{4 z_0^{3/2}} \right) + \frac{R_1}{2 \sqrt{\pi} z_0^{1/4} H_0(z_0)}. \tag{96}$$

Hence, from the upper bound on R_1 in Lemma 4, the lower bound on $H_0(z_0)$ in Corollary 1, and the bound in Lemma 2, the Lemma follows. ∎

Remark 3. *It is to be noted that for large* $|z_0|$, $\hat{C}_1 = O(1)$, *since it is clear from* (50) *and* (77) *that* $\epsilon_{1,0} = O(|z_0|^{-3/2})$ *and* $k_1 = O(|z_0|^{-3})$.

Lemma 8. *$\mathcal{N}(k)$ in (32) may be also expressed as*

$$\mathcal{N}(k) = \frac{i\Lambda\alpha^2}{\nu\hat{I}_1}\left(\frac{1+E_1}{1+E_2}\right) , \tag{97}$$

where

$$E_1 = -\frac{B_2 I_2}{B_1 I_4} + \frac{B_2 B_1^{-1} I_1 - I_3}{I_4} e^{2z_0^{3/2}} , \qquad E_2 = -\frac{I_2 I_3}{I_4 I_1} , \tag{98}$$

and have exponential bounds in α as follows

$$\left|E_1\right| \le e^{-2\alpha} J\left(\frac{R}{\alpha}\right)\left(\frac{C_{I,1}}{C_{m,4}} + \frac{2\alpha C_0}{C_{m,4}}\right) , \tag{99}$$

$$\left|E_2\right| \le \frac{C_0^2}{C_{m,4}C_{m,1}}\alpha^2 e^{-2\alpha} J\left(\frac{R}{\alpha}\right). \tag{100}$$

Proof. Dividing the numerator of (32) by $\lambda_2 e^{\alpha} B_1 I_4$, and the denominator by $-\lambda_1\lambda_2 I_1 I_4$, and noting the definitions of λ_1 and λ_2,

$$-\frac{\lambda_2\lambda_2 I_1 I_4}{\lambda_2 e^{\alpha} B_1 I_4} = \frac{\hat{I}_1}{2\alpha} , \tag{101}$$

we obtain (97), with E_1, E_2, $\hat{I}_1$ as defined above. To determine bounds, we observe that $H_0(z)$ is real valued for $z \in \mathbb{R}$ and thus has complex conjugate symmetries, and that $z_2 = z_0^*$, with $z_0^{3/2}, z_2^{3/2} \in i\mathbb{R}$, and

$$\left|B_2\right| = \left|\mathrm{Ai(z_2)}\right| = \left|\mathrm{Ai(z_0)}\right| = \left|\mathrm{B_1}\right|. \tag{102}$$

Also, it is clear from upper bounds on I_3 and lower bounds on I_4 that

$$\left|\frac{I_3}{I_4}\right| \le \left|e^{2z_2^{1/2}z_3}\right| C_{m,4}^{-1} |z_0|^{3/4} C_0 R^{-1/2} J\left(\frac{R}{\alpha}\right) = e^{-2\alpha} C_0 C_{m,4}^{-1} \alpha J\left(\frac{R}{\alpha}\right), \tag{103}$$

and that the same bound applies to $\frac{I_2}{I_4}$. We also note that

$$\left|\frac{I_1}{I_4}\right| \le \left|e^{2z_2^{1/2}z_3}\right| C_{m,4}^{-1} |z_0|^{3/4} C_0 |z_0|^{-3/4} e^{-\Re g_2(1)} = e^{-2\alpha} C_0 C_{m,4}^{-1} J\left(\frac{R}{\alpha}\right). \tag{104}$$

Combining, we get the upper bound for E_1. For E_2 we use lower bounds on I_4 and I_1 from Lemmas 5 and 4 and combine with upper bounds on I_2, I_3 in Lemma 6 to find

$$\left|E_2\right| = \left|\frac{I_2 I_3}{I_4 I_1}\right| \le \frac{C_0^2 \alpha^2}{C_{m,4}C_{m,1}} J\left(\frac{R}{\alpha}\right) e^{-2\alpha}. \tag{105}$$

∎

Proof of Proposition 1 The stated proposition follows from Lemma 8, if we define

$$E_A = \left(1 - \frac{1}{4z_0^{3/2}}\right)^{-1}\left(\hat{I}_1 - 1 + \frac{1}{4z_0^{3/2}}\right), \tag{106}$$

and the bounds on E_A as stated in Lemma 7. The exponential bound for E_1, E_2 is obvious in Lemma 8. For E_A, from estimates in (7), we only have a bound that decays with $|z_0|^{-3}$. Since all the constants are monotonically decreasing with $|z_0| = R^{-2/3}\alpha^{4/3}$ and $J(R\alpha^{-1}) \le J(1/\sqrt{3})$, it follows that we can calculate bounds in the regime $\alpha = k\sqrt{\nu} \ge \max\left\{\sqrt{3}R, \alpha_r\right\}$, precisely by evaluation at $\alpha = \alpha_r$, $\frac{R}{\alpha} = \frac{1}{\sqrt{3}}$ which results in the quoted values.

3 Behaviour of bifurcation point for $R >> 1$, $\nu << 1$, in the regime $R\nu^{1/2} >> 1$.

We denote $\mathcal{N}_{1/2}[k]$ as the evaluation of $\mathcal{N}[k]$ for $\Lambda = 1/2$, in which case $\mathcal{N}[k] = 2\Lambda\mathcal{N}_{1/2}[k]$. We require the asymptotics of $\mathcal{N}_{1/2}[k]$ for fixed k large R, small ν in the regime stated. We recall that

$$\mathcal{N}_{1/2}[k] = -\frac{i}{4\nu}\left(\frac{\alpha n(\alpha)}{D(\alpha)}\right) \tag{107}$$

where $D(\alpha)$ and $n(\alpha)$ are defined in terms of integrals of Airy functions given in §2 in the ESM. Now, with the restriction given it is easy to note that z_0, z_2 defined in (20) are each small, since $\alpha = k\sqrt{\nu}$ is small; however, z_1, z_3 are large since each is clearly $O\left(R^{1/3}\nu^{1/6}\right)$. We also note that $\arg z_1 \sim \frac{\pi}{6}$, $\arg z_3 \sim \frac{5}{6}\pi$, and the Airy function $\mathrm{Ai}(z)$ is exponentially small near z_1 and exponentially large near z_3. Furthermore, rewriting

$$I_{1,2}(z) = \int_0^\infty e^{\mp {z_0}^{1/2} z}\mathrm{Ai}(\mathrm{z})dz + \int_{z_0}^0 e^{\mp {z_0}^{1/2} z}\mathrm{Ai}(\mathrm{z})dz + \int_\infty^{z_1} e^{\mp {z_0}^{1/2} z}\mathrm{Ai}(\mathrm{z})dz, \tag{108}$$

it is clear that the last integral gives an exponentially small contribution and the leading two-order contribution comes from the first integral so that we have

$$I_{1,2} = \frac{1}{3} \mp \frac{3^{1/6}}{2\pi}\Gamma\left(\frac{2}{3}\right) z_0^{1/2} + O(z_0). \tag{109}$$

It follows that

$$\frac{I_2}{I_1} = 1 + 6\hat{a}_1 z_0^{1/2} + O(z_0)\ , \quad \text{where} \quad \hat{a}_1 = \frac{3^{1/6}}{2\pi}\Gamma\left(\frac{2}{3}\right). \tag{110}$$

On the other hand because of exponentially large behaviour at z_3 of the integrands for I_3 and I_4 in (27), on integrating the known leading order asymptotics of $\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}z^{1/4}}e^{-2/3z^{3/2}}$, we find

$$\omega I_{3,4} = -\frac{1}{2\sqrt{\pi}z_3^{3/4}} e^{-\frac{2}{3}z_3^{3/2}}\left(1 \mp \alpha + O\left(\alpha^2, z_3^{-1}\right)\right), \tag{111}$$

where we used $z_2^{1/2} z_3 = -\alpha << 1$. Therefore, it follows that

$$\frac{I_3}{I_4} = 1 - 2\alpha + O(\alpha^2). \tag{112}$$

We also have $\alpha e^{-\alpha}\lambda_1 = z_0^{1/2} e^{z_0^{3/2}}$ and $\alpha e^{\alpha}\lambda_2 = z_0^{1/2} e^{-z_0^{3/2}}$, hence we may write

$$\begin{aligned} \frac{\alpha n(\alpha)}{D(\alpha)} &\sim \frac{B_1 z_0^{1/2}}{I_1}\frac{\left(e^{-z_0^{3/2}} - e^{z_0^{3/2}}\frac{I_3}{I_4}\right)}{z_0\alpha^{-2}\left(\frac{I_3 I_2}{I_4 I_1} - 1\right)} \\ &\sim -\frac{B_1\alpha^2}{z_0^{1/2} I_1} \sim -\frac{3B_1\alpha^2}{z_0^{1/2}} = -\frac{3^{1/3}}{\Gamma\left(\frac{2}{3}\right)} e^{i\pi/6}\alpha^{4/3}R^{1/3}. \end{aligned} \tag{113}$$

Therefore, it follows that

$$\mathcal{N}_{1/2}[k] = -\frac{3^{1/3}e^{-i\pi/3}}{4\nu^{1/3}\Gamma\left(\frac{2}{3}\right)}k^{4/3}R^{1/3}\left(1 + O\left(\nu^{1/3}R^{-1/3}, \nu^{1/2}\right)\right). \tag{114}$$

Considering the bifurcation point

$$-2\Lambda_b \Re\left\{\mathcal{N}_{1/2}[k]\right\} = \nu k^4, \tag{115}$$

it follows that for fixed k we have the asymptotic balance

$$\frac{\Lambda_b 3^{1/3}}{4\nu^{1/3}\Gamma\left(\frac{2}{3}\right)} k^{4/3} R^{1/3} = \nu k^4 \tag{116}$$

implying that Λ_b scales as $\nu^{4/3} R^{-1/3}$, whereas

$$C_b = 2k^{-1}\Lambda_b \Im\left\{\mathcal{N}_{1/2}[k]\right\}, \tag{117}$$

which implies that C_b scales as ν, but is independent of R to the leading order.

4 Additional quasi-solutions and checking conditions of Theorem1 from the main part.

4.1 Quasi-solution for $k = 1$ branch for $\Lambda = 1$, $R = 20$ and $\nu = \frac{1}{10}$ and details

We chose quasi-solution $\left(C_0, \left\{\hat{H}_0(k)\right\}_{k=1}^{8}\right)$, given by expressed as rationals so as to avoid any round off errors in the computation, given by

$$\left[\frac{8554}{1397}, -\frac{12885}{23828}, -\frac{1043}{4331} - \frac{435\,i}{2339}, \frac{1409}{55585} - \frac{585\,i}{7199}, \frac{302}{18357} - \frac{127\,i}{41559}, \frac{30}{16099} + \frac{91\,i}{36906}, -\frac{36}{152065} + \frac{77\,i}{168821}, -\frac{9}{111589} - \frac{3\,i}{1407007}, -\frac{4}{800731} - \frac{13\,i}{1170328}\right] \tag{118}$$

with corresponding $\left\{\mathcal{N}_{1/2}[k]\right\}_{k=1}^{8}$ obtained from integrals of Airy function, obtained with the help of symbolic manipulation tool and expressed as rational numbers

$$\left[-\frac{21061}{126378} + \frac{88807\,i}{27831}, -\frac{35583}{53450} + \frac{49758\,i}{7591}, -\frac{126496}{84899} + \frac{107673\,i}{10486}, -\frac{24107}{9219} + \frac{88338\,i}{6089}, -\frac{56973}{14279} + \frac{394296\,i}{20285}, -\frac{109483}{19766} + \frac{142793\,i}{5668}, -\frac{76036}{10601} + \frac{203838\,i}{6397}, -\frac{67113}{7621} + \frac{1427719\,i}{36144}\right] \tag{119}$$

and with choice $K = 8$, with help of symbolic computational tools, it is easy to check that

$$\begin{gathered}\epsilon_R \le 2.416 \times 10^{-6}, \quad \|\hat{H}_0\|_{l^1} \le 0.9506\ , \quad M_g \le 2.8703, \quad \epsilon_u \le 0.014136\ , \\ \epsilon_q \le 2.4785 \times 10^{-7}\ , \quad C_L \le 2.7284\ , \quad \gamma_{1,1} \le 8.450\ , \quad \beta_{1,1} \le 12.623\ , \quad \beta_{1,2} \le 34.933\ , \\ \beta_{2,1} \le 0.18099 \quad \beta_{2,2} \le 1.51523\ , \quad M_{\mathcal{L}} \le 36.45\ , \quad \epsilon \le 0.8803 \times 10^{-4}\ , \quad \beta_c \le 0.037\end{gathered} \tag{120}$$

implying that the condition for application of Theorem 1 in the main part is satisfied and hence there exists solution $(C, \hat{H})$ near quasi-solution $(C_0, \hat{H}_0)$ with

$$\left|C - C_0\right| + \|\hat{H} - \hat{H}_0\|_{l^1} \le 2\epsilon \le 1.7606 \times 10^{-4} \tag{121}$$

4.2 Quasi-solution for $k = 1$ branch for $\Lambda = \frac{6}{5}$, $R = 50$, $\nu = \frac{1}{10}$ and details

For quasi-solution $\left(C_0, \left\{\hat{H}_0(k)\right\}_{k=1}^{12}\right)$, given by expressed as rationals so as to avoid any round off errors in the computation, given by

$$\left[\frac{52299}{10060}, -\frac{34717}{16727}, -\frac{8178}{5321} - \frac{26965\,i}{98492}, -\frac{23247}{26941} - \frac{21767\,i}{35052}, -\frac{14284}{105875} - \frac{39780\,i}{68137}, \frac{9473}{68179} - \frac{8061\,i}{36734},\right.$$
$$\frac{3386}{35205} - \frac{6982\,i}{254943}, \frac{12791}{374382} + \frac{4099\,i}{261327}, \frac{5668}{1013251} + \frac{2758\,i}{218663}, -\frac{1580}{1035709} + \frac{2639\,i}{568600}, -\frac{1470}{1009301} + \frac{1113\,i}{1339817},$$
$$\left. -\frac{1427}{2590444} - \frac{69\,i}{577513}, -\frac{434}{3969305} - \frac{161\,i}{1064144}\right] \qquad (122)$$

with corresponding $\left\{\mathcal{N}_{1/2}[k]\right\}_{k=1}^{12}$ obtained from integrals of Airy function, obtained with the help of symbolic manipulation tool and expressed as rational numbers

$$\left[-\frac{34597}{84279} + \frac{19739\,i}{6112}, -\frac{79652}{50645} + \frac{17607\,i}{2573}, -\frac{94039}{28408} + \frac{77003\,i}{6914}, -\frac{4341}{802} + \frac{9232\,i}{567}, -\frac{96921}{12583} + \frac{121883\,i}{5456},\right.$$
$$-\frac{20407}{2028} + \frac{291718\,i}{9953}, -\frac{39064}{3143} + \frac{165167\,i}{4443}, -\frac{190290}{12883} + \frac{315821\,i}{6878}, -\frac{113933}{6673} + \frac{8718\,i}{157},$$
$$\left. -\frac{353192}{18271} + \frac{324679\,i}{4919}, -\frac{109469}{5083} + \frac{256797\,i}{3320}, -\frac{160351}{6770} + \frac{380017\,i}{4243}\right] \qquad (123)$$

and with choice $K = 12$, with help of symbolic computational tools, it is easy to check that

$$\begin{aligned} &\epsilon_R \leq 9.316 \times 10^{-5}, \quad \|\hat{H}_0\|_{l^1} \leq 5.7174\ , \quad M_g \leq 0.37019, \quad \epsilon_u \leq 0.004643\ , \\ &\epsilon_q \leq 1.1076 \times 10^{-6}\ , \quad C_L \leq 2.1165\ , \quad \gamma_{1,1} \leq 12.393\ , \quad \beta_{1,1} \leq 14.124\ , \quad \beta_{1,2} \leq 30.025\ , \\ &\beta_{2,1} \leq 0.06587 \quad \beta_{2,2} \leq 1.1448\ , \quad M_{\mathcal{L}} \leq 31.17\ , \quad \epsilon \leq 2.91 \times 10^{-3}\ , \quad \beta_c \leq 0.13402 \end{aligned} \qquad (124)$$

implying that condition for application of Theorem 1 in the main part is satisfied and hence there exists solution $(C, \hat{H})$ near quasi-solution $(C_0, \hat{H}_0)$ with

$$\left|C - C_0\right| + \|\hat{H} - \hat{H}_0\|_{l^1} \leq 2\epsilon \leq 5.82 \times 10^{-3} \qquad (125)$$

4.3 Quasi-solution for $k = 1$ branch $\Lambda = \frac{6}{5}$, $R = 100$ and $\nu = \frac{1}{10}$

We chose a quasi-solution was $\left(C_0, \left\{\hat{H}_0(k)\right\}_{k=1}^{20}\right)$, given by

$$\left[\frac{48637}{20794}, -\frac{58399}{15282}, -\frac{45295}{15473} - \frac{16699\,i}{36526}, -\frac{102139}{52823} - \frac{12263\,i}{11941}, -\frac{57595}{70024} - \frac{32714\,i}{25373}, \frac{10345}{114736} - \frac{34465\,i}{36889}, \frac{33251}{94515} - \frac{16258\,i}{47621},\right.$$
$$\frac{12923}{55709} - \frac{5077\,i}{213589}, \frac{18739}{202756} + \frac{6232\,i}{112919}, \frac{4753}{252538} + \frac{4416\,i}{97519}, -\frac{2043}{370751} + \frac{1415\,i}{66602}, -\frac{2453}{323823} + \frac{1090\,i}{184447}, -\frac{2074}{503431} + \frac{152\,i}{5441943},$$
$$-\frac{1586}{1143277} - \frac{1313\,i}{1251721}, -\frac{223}{1150297} - \frac{775\,i}{1108116}, \frac{267}{2384990} - \frac{473\,i}{1699429}, \frac{288}{2723219} - \frac{841\,i}{13412396}, \frac{993}{20003308} + \frac{71\,i}{12032325},$$
$$\left. \frac{217}{15003431} + \frac{121\,i}{8595382}, \frac{25}{21306754} + \frac{71\,i}{8861216}, -\frac{47}{29899294} + \frac{42\,i}{14877883}\right] \qquad (126)$$

The corresponding $\{\mathcal{N}_{1/2}[k]\}_{k=1}^{20}$ obtained from integral of Airy function was

$$\Big[-\frac{24633}{31499}+\frac{58771\,i}{17520},-\frac{40991}{15181}+\frac{98653\,i}{13005},-\frac{29317}{5703}+\frac{162055\,i}{12583},-\frac{238280}{30497}+\frac{33202\,i}{1735},-\frac{53574}{5041}+\frac{166797\,i}{6346},$$
$$-\frac{90446}{6673}+\frac{100423\,i}{2928},-\frac{73094}{4411}+\frac{165113\,i}{3824},-\frac{179017}{9108}+\frac{122002\,i}{2305},-\frac{136517}{5992}+\frac{198470\,i}{3123},-\frac{14861}{573}+\frac{379703\,i}{5060},$$
$$-\frac{263396}{9053}+\frac{159403\,i}{1824},-\frac{311356}{9655}+\frac{1044159\,i}{10379},-\frac{174657}{4936}+\frac{402611\,i}{3511},-\frac{407615}{10589}+\frac{316605\,i}{2443},$$
$$-\frac{439061}{10562}+\frac{1193144\,i}{8207},-\frac{182209}{4085}+\frac{168671\,i}{1041},-\frac{305443}{6418}+\frac{430539\,i}{2398},-\frac{64067}{1268}+\frac{249188\,i}{1259},$$
$$-\frac{206295}{3863}+\frac{274307\,i}{1263},-\frac{114909}{2044}+\frac{321584\,i}{1355}\Big] \tag{127}$$

With choice $K = 20$, with help of symbolic computational tools, it is easy to check that

$$\begin{gathered}\epsilon_R \leq 2.1521\times 10^{-6}, \quad \|\hat{H}_0\|_{l^1} \leq 12.361\ , \quad M_g \leq 0.1672, \quad \epsilon_u \leq 0.00109\ , \\ \epsilon_q \leq 4.072\times 10^{-9}\ , \quad C_L \leq 2.0664\ , \quad \gamma_{1,1} \leq 13.746\ , \quad \beta_{1,1} \leq 14.185\ , \quad \beta_{1,2} \leq 29.342\ , \\ \beta_{2,1} \leq 0.154385 \quad \beta_{2,2} \leq 1.03303\ , \quad M_{\mathcal{L}} \leq 30.3742\ , \quad \epsilon \leq 6.54\times 10^{-5}\ , \quad \beta_c \leq 0.00133\end{gathered} \tag{128}$$

implying that condition for application of Theorem 1 in the main part is satisfied and hence there exists solution $(C, \hat{H})$ near quasi-solution $(C_0, \hat{H}_0)$ with

$$\left|C - C_0\right| + \|\hat{H} - \hat{H}_0\|_{l^1} \leq 2\epsilon \leq 1.308\times 10^{-4} \tag{129}$$

5 Computed travelling wave profiles

Here we give results of the computed wave profiles corresponding to the results of Figures 2 and 3. This is done for all marked points on each solution branch where existence of solutions was proved. In all the results shown we depict linearly stable solutions with a blue colour and unstable ones are coloured red. This way the reader can follow the bifurcations that take place along individual branches as Λ increases.

5.1 Wave profiles for $\nu = 1/10$ and different R and Λ

Results are shown in Figures 1-3 corresponding to $R = 20$, 50 and 100, respectively. The left panels show branch 1 $k = 1$ solutions, and the right panels the corresponding branch 2 solutions. This is clear from the figures because the former are $2\pi-$periodic and the latter are $\pi-$periodic.

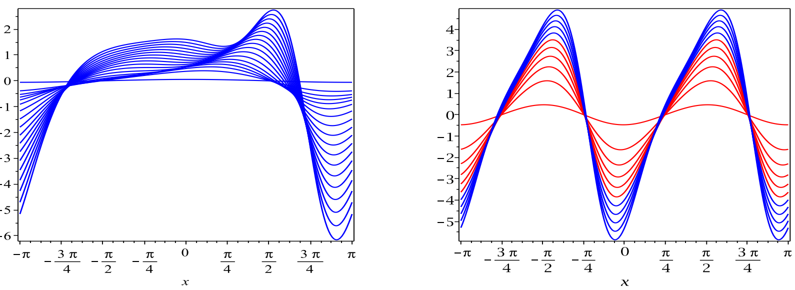

Figure 1: $H_0(x)$ vs. x for $R = 20$, $\nu = 1/10$. Left: Branch 1, $\Lambda = 0.302, 0.4, 0.5, 0.6, \cdots 2.0$. Right: Branch 2, $\Lambda = 1.21, 1.3, 1.4, \cdots 2.2$. Blue - stable; Red - unstable.

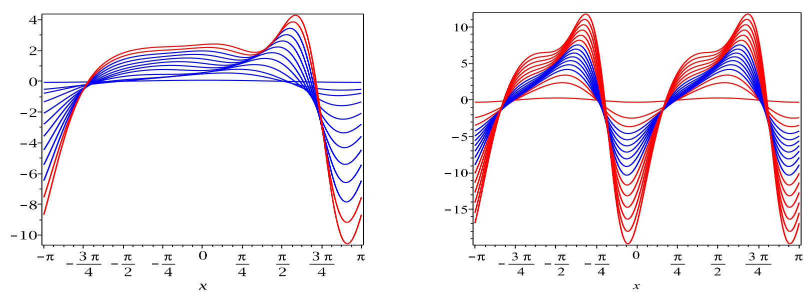

Figure 2: $H_0(x)$ vs. x for $R = 50$, $\nu = 1/10$. Left: Branch 1, $\Lambda = 0.123, 0.2, 0.3, 0.6, \cdots 1.2$. Right: Branch 2, $\Lambda = 0.51, 0.6, 0.7, 0.8, \cdots 2.0$. Blue - stable; Red - unstable.

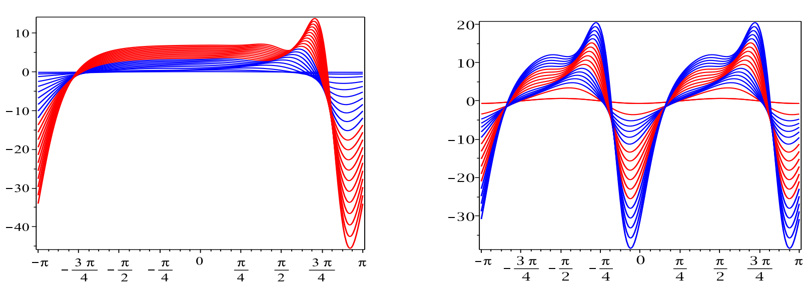

Figure 3: $H_0(x)$ vs. x for $R = 100$, $\nu = 1/10$. Left: Branch 1, $\Lambda = 0.065, 0.1, 0.2, 0.3, \cdots 2.0$. Right: Branch 2, $\Lambda = 0.3, 0.6, 0.7, 0.8, \cdots 2.0$. Blue - stable; Red - unstable.

5.2 Wave profiles for $\nu = 1/20$ and different R and Λ

Results are shown in Figures 4-6 corresponding to $R = 20$, 50 and 100, respectively. The left panels show branch 1 $k = 1$ solutions, and the right panels the corresponding branch 2 solutions. This is clear from the figures because the former are $2\pi-$periodic and the latter are $\pi-$periodic.

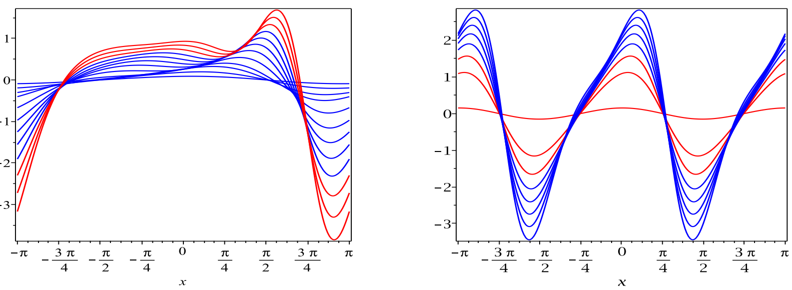

Figure 4: $H_0(x)$ vs. x for $R = 20$, $\nu = 1/20$. Left: Branch 1, $\Lambda = 0.160, 0.2, 0.3, 0.6, \cdots 1.2$. Right: Branch 2, $\Lambda = 0.602, 0.7, 0.8, \cdots 1.3$. Blue - stable; Red - unstable.

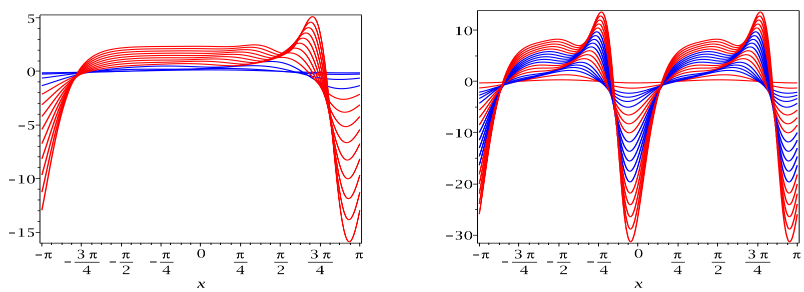

Figure 5: $H_0(x)$ vs. x for $R = 50$, $\nu = 1/20$. Left: Branch 1, $\Lambda = 0.07, 0.1, 0.2, 0.3, \cdots 1.2$. Right: Branch 2, $\Lambda = 0.25, 0.3, 0.4, \cdots 2.0$. Blue - stable; Red - unstable.

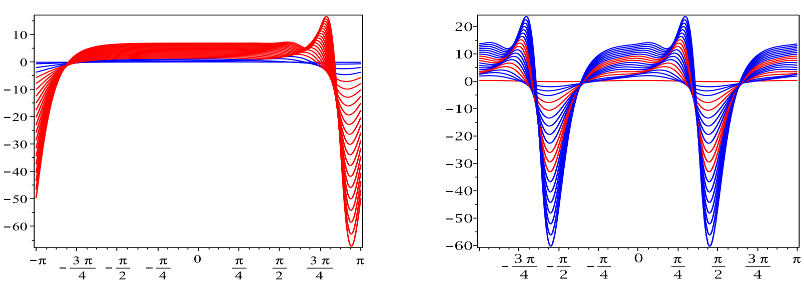

Figure 6: $H_0(x)$ vs. x for $R = 100$, $\nu = 1/20$. Left: Branch 1, $\Lambda = 0.032, 0.1, 0.2, 0.3, \cdots 2.0$. Right: Branch 2, $\Lambda = 0.136, 0.2, 0.3, 0.4, \cdots 2.0$. Blue - stable; Red - unstable.